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Flows on quaternionic-Kähler and very special real manifolds

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Abstract

BPS solutions of 5-dimensional supergravity correspond to certain gradient flows on the product $M \times N$ of a quaternionic-Kähler manifold M of negative scalar curvature and a very special real manifold N of dimension $n \geq 0$. Such gradient flows are generated by the "energy function" $f = P^2$, where P is a (bundle-valued) moment map associated to $n+1$ Killing vector fields on M. We calculate the Hessian of f at critical points and derive some properties of its spectrum for general quaternionic-Kähler manifolds. For the homogeneous quaternionic-Kähler manifolds we prove more specific results depending on the structure of the isotropy group. For example, we show that there always exists a Killing vector field vanishing at a point $p \in M$ such that the Hessian of f at p has split signature. This generalizes results obtained recently for the complex hyperbolic plane (universal hypermultiplet) in the context of 5-dimensional supergravity. For symmetric quaternionic-Kähler manifolds we show the existence of non-degenerate local extrema of f, for appropriate Killing vector fields. On the other hand, for the non-symmetric homogeneous quaternionic-Kähler manifolds we find degenerate local minima.

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Contents

1 Introduction

Theories of 5 dimensional supergravity have recently obtained increased attention in the context of the AdS/CFT correspondence (for a review, see [1]) and for a supersymmetrisation of the Randall–Sundrum (RS) scenario [2, 3]. In both cases one eventually uses a 5-dimensional metric of the form

$$\mathrm{d}s^2 = a(x^5)^2 \mathrm{d}x^{\underline{\mu}} \mathrm{d}x^{\underline{\nu}} \eta_{\underline{\mu\nu}} + (\mathrm{d}x^5)^2 \,, \tag{1.1}$$

where $\underline{\mu}, \underline{\nu} = 0, 1, 2, 3$. We thus have a flat 4-dimensional space with a warp factor a that depends on the fifth coordinate x^5. The warp factor is interpreted as the energy scale of the renormalization group flow in the comparison between 5-dimensional AdS theories and 4-dimensional conformal theories. For this application it should therefore run from a low value (infrared: IR) to a high value (ultraviolet: UV). For the RS scenario it should have a maximum at the value of x^5 which we want to associate with the position of a domain wall, and should drop off at both infinities $x^5 = \pm\infty$ towards zero. We are then considering a scenario with one domain wall, which has been called 'smooth'. This essentially means that the configuration is a solution of the field equations of a 5-dimensional (matter-coupled) supergravity theory without extra sources (which would be singular insertions of a brane in the bulk theory).

Supersymmetric theories in 5 dimensions with the minimal number of supersymmetries (8 real supercharges) have scalars which occur in vector multiplets and in hypermultiplets. Other multiplets, like tensor multiplets, could be added, but would not change anything below. The kinetic terms of the scalars define a metric on the target manifold $M \times N$

that is a direct product of a quaternionic-Kähler manifold M of dimension $4r$ of negative scalar curvature[1] parametrized by the scalars of the r hypermultiplets and a very special real manifold N of dimension n, parametrized by the scalars of the n vector multiplets [4, 5]. The general actions have been conveniently written down in [6]. We will recall the notion of quaternionic-Kähler manifolds in section 2 and of very special real manifolds in section 4.

For the above-mentioned applications, one has to look for supergravity solutions for which the only non-zero fields are the scalars and the warp factor a in (1.1), and these depend only on x^5. The kinematics is then determined by the kinetic terms, encoded in the geometry of the target manifold, and by the scalar potential. In supersymmetric theories with 8 or more real supercharges the potential is determined by the gauging of (infinitesimal) isometries of the manifold. An isometry can be gauged if there is a vector in the theory that can serve as a connection. The theory contains $n+1$ vectors if n is the dimension of the very special real manifold. Indeed, pure supergravity contains already 1 vector, the 'graviphoton', while the other n originate from the vector multiplets. For every isometry there is a moment map, as we shall recall in section 2. When we gauge $n+1$ of these, the potential depends on the 'dressed moment map' (see section 5), which is a linear combination of $n+1$ moment maps on M with functions on N as coefficients. This is an $\mathfrak{sp}(1) = \mathfrak{su}(2)$ triplet P^α. The scalars of supersymmetry-preserving (BPS) solutions of the theory have to take values in the submanifold determined by the condition [7]

$$\frac{\partial}{\partial \phi^x}\left(\frac{P^\alpha}{|P|}\right) = 0\,, \tag{1.2}$$

where ϕ^x, with $x = 1, \ldots, n$, are the coordinates of N, and $\alpha = 1, 2, 3$. Under this condition[2], the scalar potential depends only on the 'energy function' [11, 7]

$$f = \tfrac{3}{2} W^2 = P^\alpha P^\alpha\,. \tag{1.3}$$

The solutions are then determined by corresponding 'flow equations' which determine the scalars as functions of x^5. These equations are (with prime denoting the derivative with respect to x^5)

$$\frac{a'}{a} = \pm W\,, \qquad \phi^{x\prime} = \mp 3 g^{xy} \frac{\partial}{\partial \phi^y} W\,, \qquad q^{X\prime} = \mp 3 g^{XY} \frac{\partial}{\partial q^Y} W\,, \tag{1.4}$$

where W is the positive root in (1.3), and the $\pm$ sign can be chosen according to the sign of a'/a, but then has to be used consistently in the other equations. The sign can flip when W reaches a zero. Analogous to the coordinates ϕ^x for the real manifold, we see here the coordinates of the quaternionic-Kähler manifold q^X (scalars of the hypermultiplet), where $X = 1, \ldots, 4r$ for r the quaternionic dimension of M. These equations thus determine a gradient line on the product of the manifolds N and M parametrized by x^5, which runs from $-\infty$ to ∞.

[1]We consider local supersymmetry. For rigid supersymmetry, the scalar curvature of M would be zero, implying that it is (locally) a hyper-Kähler manifold. Furthermore, we consider here always theories in a 5-dimensional space with Minkowski signature.

[2]This condition can be relaxed if the 4-dimensional part of the metric is generalized from the flat one in (1.1) to a curved one [8, 9, 10].

The equations imply that $(\ln a)'' \leq 0$. The essential properties of the flow can therefore be seen by analyzing fixed points of the gradient flow and zeros of W. The former are the stationary values of the scalars. We look for solutions that have 'fixed points' at $x^5 = \pm\infty$. These fixed points can be found from algebraic equations [7].

The behaviour of a solution near a fixed point p is determined by whether W increases or decreases when we approach p in a certain direction. This can be read off from the $(n + 4r) \times (n + 4r)$ matrix

$$\mathcal{U}_\Sigma{}^\Lambda \equiv \frac{3}{W} g^{\Lambda\Xi} \partial_\Sigma \partial_\Xi W \bigg|_{\partial W=0} = \frac{3}{2f} g^{\Lambda\Xi} \partial_\Sigma \partial_\Xi f \bigg|_{\partial f=0} , \tag{1.5}$$

where Λ, Σ enumerate all the scalars, and thus ∂_Λ contains derivatives with respect to ϕ^x and q^X. If the flow is along a direction corresponding to a positive part of this matrix, the scalars flow to this fixed point with large values of the warp factor a, and the point is called a UV attractor. If the flow is along a direction where the matrix $\mathcal{U}$ is negative, the scalars are attracted to this point for small values of a, and this point is called an IR attractor or IR fixed point (see e.g. [12, 7] for more details). The eigenvalues of $\mathcal{U}$ are also the conformal weights of the corresponding operators in the conformal dual to the supergravity theory.

The purpose of this paper is to derive general properties of such flows. These are mostly determined from knowledge of the matrix $\mathcal{U}$. Our main result is a suitable formula for this matrix. Furthermore we can derive general results on the possibility of UV and IR critical points in symmetric or homogeneous quaternionic-Kähler manifolds.

The paper is organized as follows. In section 2 we recall the basis properties of quaternionic-Kähler manifolds and the moment map. We then analyse the part of the matrix $\mathcal{U}$ (Hessian of the energy) for a pure quaternionic-Kähler manifold in section 3, and derive properties of attractor points. The very special real manifolds are introduced in section 4 and the adapted moment map in section 5. This allows us to find the properties of the full Hessian in section 6. Finally we give conclusions in section 7. An effort is made to translate mathematical formulae in notation readable to physicists and vice versa. In particular, we have given a presentation of very special real geometry which is accessible to mathematicians. Some remarks on our notation are gathered in the appendix.

2 Quaternionic-Kähler moment map

We start by recalling the notion of quaternionic-Kähler manifold. Let (M, g) be a Riemannian manifold. A **quaternionic-Kähler structure** Q on M is a rank 3 subbundle $Q \subset \mathrm{End}(TM)$ invariant under parallel transport such that locally $Q = \mathrm{span}\{J_1, J_2, J_3 = J_1 J_2\}$, where the J_α are locally defined skew-symmetric almost complex structures on M.

With respect to local coordinates q^X, with $X = 1, \ldots 4r = \dim M$, the almost complex structures J_α have components $J_{\alpha X}{}^Y$ satisfying[3]

$$J_{\alpha X}{}^Y J_{\beta Y}{}^Z = -\delta_{\alpha\beta} \delta_X{}^Z + \varepsilon_{\alpha\beta\gamma} J_{\gamma X}{}^Z , \tag{2.1}$$

[3]Here and below, a sum over repeated indices is understood.

where $\varepsilon_{\alpha\beta\gamma}$ is completely antisymmetric with ε_{123}=1. The invariance of Q under the Levi-Civita connection ∇ is tantamount to the existence of a triplet of one-forms ω_α such that

$$\nabla J_\alpha = -2\varepsilon_{\alpha\beta\gamma}\omega_\beta J_\gamma\,. \tag{2.2}$$

The ω_α may be determined from this equation, which can be rewritten as the full covariant constancy of the complex structures; in components:

$$\mathcal{D}_Z J_{\alpha X}{}^Y := \partial_Z J_{\alpha X}{}^Y - \Gamma^W_{ZX} J_{\alpha W}{}^Y + \Gamma^Y_{ZW} J_{\alpha X}{}^W + 2\varepsilon_{\alpha\beta\gamma}\omega_{\beta Z} J_{\gamma X}{}^Y = 0\,. \tag{2.3}$$

Here Γ are the Christoffel symbols of ∇ and $2\varepsilon_{\alpha\beta\gamma}\omega_\beta$ is the connection matrix of the connection induced by ∇ in the bundle Q. We note that $\mathcal{D}$ is here a connection in $Q\otimes\mathbb{R}^3$ induced by the Levi-Civita connection on Q and the connection on the trivial rank 3 bundle over M defined by $\mathcal{D}e_\alpha = 2\varepsilon_{\alpha\beta\gamma}\omega_\beta e_\gamma$, where e_α is the standard basis of $\mathbb{R}^3$. Note that equation (2.3) means that the section $J := J_\alpha \otimes e_\alpha$ is parallel with respect to $\mathcal{D}$.

In general, $\mathcal{D}$ contains the gauge field of all the transformations of the object on which it acts (see appendix).

A Riemannian manifold admits a quaternionic-Kähler structure if and only if its holonomy group is a subgroup of Sp(r)Sp(1), where $4r = \dim M$. The group Sp(r)Sp(1) is the linear group normalizing a quaternionic structure on $\mathbb{R}^{4r}$ and preserving a compatible Euclidean scalar product.

A **quaternionic-Kähler manifold** of $\dim M = 4r > 4$ is a Riemannian manifold endowed with a quaternionic-Kähler structure. In dimension 4 (the case $r = 1$) this definition would correspond simply to the notion of oriented Riemannian 4-fold. Instead we will assume in addition that Q annihilates the curvature tensor of the manifold (M, g), i.e.

$$J_{\alpha X}{}^V R_{VYZW} + J_{\alpha Y}{}^V R_{XVZW} + J_{\alpha Z}{}^V R_{XYVW} + J_{\alpha W}{}^V R_{XYZV} = 0\,. \tag{2.4}$$

This condition is automatically satisfied if $r > 1$. Then the following result holds in all dimensions [13].

Theorem 1 *The curvature tensor R of a quaternionic-Kähler manifold of dimension $4r$ is of the form*

$$R = \nu R_0 + \mathcal{W}\,,$$

where R_0 is the curvature tensor of the quaternionic projective space, $\nu = \frac{scal}{4r(r+2)}$ is the reduced scalar curvature and $\mathcal{W}$ is an algebraic curvature tensor of type $\mathfrak{sp}(r)$ (the "Weyl curvature").

This means that the components can be written as

$$\begin{aligned} R_{XYZW} \;&=\; \nu\left[\tfrac{1}{2} g_{Z[X} g_{Y]W} + \tfrac{1}{2} J^\alpha_{XY} J^\alpha_{ZW} - \tfrac{1}{2} J^\alpha_{Z[X} J^\alpha_{Y]W}\right] \\ &\quad + f^{iA}_X f^{jB}_Y \varepsilon_{ij} f^{kC}_Z f^{\ell D}_W \varepsilon_{k\ell} \Sigma_{ABCD}\,, \end{aligned} \tag{2.5}$$

where the antisymmetrization of a tensor T_{XY} is defined as $T_{[XY]} := \frac{1}{2}(T_{XY} - T_{YX})$, the f^{iA}_X are the vielbeins of the manifold ($A = 1, \ldots, 2r$ and $i, j = 1, 2$) and Σ_{ABCD} is completely symmetric. We do not use vielbeins in this paper, but the interested reader may find a discussion of quaternionic-Kähler manifold in terms of vielbeins in the article on quaternionic-Kähler manifolds in [14] and a definition of quaternionic-Kähler manifolds that starts from vielbeins has been given in [15], see also [16, 17, 7].

To avoid confusion we emphasize that here and in all coordinate expressions we use $X, Y, \ldots$ to denote indices from $1, \ldots, 4r$, following the convention in the 5d supergravity literature. In coordinate free formulas the same letters, in sans-serif font, $\mathsf{X}, \mathsf{Y}, \ldots$, will denote vector fields.

It follows from Theorem 1 that quaternionic-Kähler manifolds are Einstein, in fact, $Ric = \nu(r+2)g$. Only in the case $\nu = 0$, can we choose the three local almost complex structures spanning the quaternionic structure Q to be parallel and, in particular, integrable. This case corresponds to locally hyper-Kählerian manifolds and is excluded in the following discussion. So from now on $\nu \neq 0$. Supergravity fixes $\nu = -1$, but we will keep ν general below.

To any quaternionic-Kähler manifold we can associate the parallel 4-form

$$\Omega := \rho_\alpha \wedge \rho_\alpha \,,$$

where the $\rho_\alpha = g(\cdot, J_\alpha \cdot)$ are the 2-forms ("Kähler forms") associated to any choice of three local almost complex structures $(J_1, J_2, J_3 = J_1 J_2)$ spanning Q (its components are just the components of the almost complex structures with indices lowered by the metric). Let K be a Killing vector field on a quaternionic-Kähler manifold (M, g, Q). Then K normalizes Q. Indeed, if (M, g) is locally symmetric, then the Lie algebra of all Killing vector fields is well known. If (M, g) is not locally symmetric and $\dim M = 4r > 4$ then the holonomy Lie algebra is $\mathfrak{hol} = \mathfrak{sp}(r) \oplus \mathfrak{sp}(1)$ and any Killing vector field normalizes the holonomy Lie algebra and in particular its $\mathfrak{sp}(1)$-factor, which defines the quaternionic-Kähler structure Q. This proves that K normalizes Q, i.e.[4] $\mathcal{L}_{\mathsf{K}} J^\alpha = b^{\alpha\beta} J^\beta$ for some $b^{\alpha\beta}(q^X)$. This amounts to

$$(\mathcal{D}_{[X} \mathsf{K}^Z) J^\alpha_{Y]Z} = -\nu \varepsilon^{\alpha\beta\gamma} J^\beta_{XY} P^\gamma \,, \tag{2.6}$$

for some $P^\gamma(q^X)$ with normalization chosen for later convenience.

The 3-form obtained from the 4-form Ω by contraction,

$$\iota_{\mathsf{K}} \Omega = \rho_\alpha(\mathsf{K}, \cdot) \wedge \rho_\alpha$$

is closed:

$$\mathrm{d}\iota_{\mathsf{K}} \Omega = \mathcal{L}_{\mathsf{K}} \Omega - \iota_{\mathsf{K}} \mathrm{d}\Omega = 0 \,.$$

Proposition 1 *The three-form $\iota_{\mathsf{K}}\Omega$ is exact*

$$\nu \iota_{\mathsf{K}} \Omega = \mathrm{d}\rho \,, \quad \rho := \langle \nabla \mathsf{K}, J_\alpha \rangle \rho_\alpha \,,$$

where $\langle \cdot, \cdot \rangle$ is the canonical scalar product on $\mathrm{End}\, TM$ *normalized such that $\langle J_\alpha, J_\beta \rangle = \delta_{\alpha\beta}$.*

[4]Here $\mathcal{L}_{\mathsf{K}}$ is the Lie derivative, i.e. $\mathcal{L}_{\mathsf{K}} \mathsf{X} = [\mathsf{K}, \mathsf{X}]$ for all vector fields X.

Proof: Let us compute $\mathrm{d}\rho = \mathrm{alt}\nabla\rho$:

$$\nabla\rho = \langle\nabla^2\mathsf{K}, J_\alpha\rangle\rho_\alpha - 2\varepsilon_{\alpha\beta\gamma}\left(\langle\nabla\mathsf{K}, \omega_\beta J_\gamma\rangle\rho_\alpha + \langle\nabla\mathsf{K}, J_\alpha\rangle\omega_\beta\otimes\rho_\gamma\right) . \tag{2.7}$$

Here, the last two terms on the right hand side cancel each other. The first term is computed using the following lemma.

Lemma 1 *Let* K *be a Killing vector field on a Riemannian manifold with curvature tensor* R*. Then the second covariant derivative of* K *is given by*[5]

$$\nabla^2\mathsf{K} = R(\cdot,\mathsf{K})\,. \tag{2.8}$$

Proof: We prove first that the tensor $\nabla^2\mathsf{Z} - R(\cdot,\mathsf{Z})$ is symmetric for any vector field Z. This follows from the Bianchi identity:

$$\nabla^2_{\mathsf{X},\mathsf{Y}}\mathsf{Z} - R(\mathsf{X},\mathsf{Z})\mathsf{Y} - (\nabla^2_{\mathsf{Y},\mathsf{X}}\mathsf{Z} - R(\mathsf{Y},\mathsf{Z})\mathsf{X}) = R(\mathsf{X},\mathsf{Y})\mathsf{Z} - R(\mathsf{X},\mathsf{Z})\mathsf{Y} + R(\mathsf{Y},\mathsf{Z})\mathsf{X} = 0\,.$$

So it is sufficient to check that $\langle\nabla^2_{\mathsf{X},\mathsf{X}}\mathsf{K},\mathsf{Y}\rangle = \langle R(\mathsf{X},\mathsf{K})\mathsf{X},\mathsf{Y}\rangle$ for all vector fields X and Y. We can assume that $[\mathsf{X},\mathsf{Y}] = 0$, since we are checking an identity between tensors; and we use the Killing equation $\langle\nabla_{\mathsf{X}}\mathsf{K},\mathsf{Y}\rangle = -\langle\nabla_{\mathsf{Y}}\mathsf{K},\mathsf{X}\rangle$.

$$\begin{aligned}
\langle\nabla^2_{\mathsf{X},\mathsf{X}}\mathsf{K},\mathsf{Y}\rangle &= \langle\nabla_{\mathsf{X}}\nabla_{\mathsf{X}}\mathsf{K} - \nabla_{\nabla_{\mathsf{X}}\mathsf{X}}\mathsf{K},\mathsf{Y}\rangle \\
&= \mathsf{X}\langle\nabla_{\mathsf{X}}\mathsf{K},\mathsf{Y}\rangle - \langle\nabla_{\mathsf{X}}\mathsf{K},\nabla_{\mathsf{X}}\mathsf{Y}\rangle + \langle\nabla_{\mathsf{Y}}\mathsf{K},\nabla_{\mathsf{X}}\mathsf{X}\rangle \\
&= -\mathsf{X}\langle\nabla_{\mathsf{Y}}\mathsf{K},\mathsf{X}\rangle - \langle\nabla_{\mathsf{X}}\mathsf{K},\nabla_{\mathsf{X}}\mathsf{Y}\rangle + \langle\nabla_{\mathsf{Y}}\mathsf{K},\nabla_{\mathsf{X}}\mathsf{X}\rangle \\
&= -\langle\nabla_{\mathsf{X}}\nabla_{\mathsf{Y}}\mathsf{K},\mathsf{X}\rangle - \langle\nabla_{\mathsf{X}}\mathsf{K},\nabla_{\mathsf{X}}\mathsf{Y}\rangle \\
&= -\langle\nabla_{\mathsf{X}}\nabla_{\mathsf{Y}}\mathsf{K},\mathsf{X}\rangle - \langle\nabla_{\mathsf{X}}\mathsf{K},\nabla_{\mathsf{Y}}\mathsf{X}\rangle \\
&= -\langle\nabla_{\mathsf{X}}\nabla_{\mathsf{Y}}\mathsf{K},\mathsf{X}\rangle + \langle\nabla_{\mathsf{Y}}\nabla_{\mathsf{X}}\mathsf{K},\mathsf{X}\rangle \\
&= -\langle R(\mathsf{X},\mathsf{Y})\mathsf{K},\mathsf{X}\rangle = \langle R(\mathsf{X},\mathsf{K})\mathsf{X},\mathsf{Y}\rangle
\end{aligned}$$

□

Now, using this lemma we obtain

$$\begin{aligned}
\nabla\rho &= \langle\nabla^2\mathsf{K}, J_\alpha\rangle\rho_\alpha = \langle R(\cdot,\mathsf{K}), J_\alpha\rangle\rho_\alpha \\
&= \tfrac{\nu}{2}\rho_\alpha(\cdot,\mathsf{K})\otimes\rho_\alpha\,.
\end{aligned} \tag{2.9}$$

Here, we use the fact that the $\mathfrak{sp}(1)$-part of $R = \nu R_0 + \mathcal{W}$ is given by the middle term of the first line of (2.5) (the other terms annihilate under multiplication with $J^{\beta XY}$):

$$R^{\mathfrak{sp}(1)} = \nu R_0^{\mathfrak{sp}(1)} = \tfrac{\nu}{2}\rho_\alpha J_\alpha\,, \tag{2.10}$$

[5] In equations as the one below, where vectors are not explicitly written, they should be understood as appearing consistently from left to right, e.g. below: $\nabla^2_{\mathsf{X},\mathsf{Y}}\mathsf{K} := \nabla_{\mathsf{X}}\nabla_{\mathsf{Y}}\mathsf{K} - \nabla_{\nabla_{\mathsf{X}}\mathsf{Y}}\mathsf{K} = R(\mathsf{X},\mathsf{K})\mathsf{Y}$. Using the covariant derivative $\mathcal{D}$ instead, this equation may be written in a coordinate basis in the form $\mathcal{D}_X\mathcal{D}_Y\mathsf{K}^Z = R^Z{}_{YXW}\mathsf{K}^W$.

see Theorem 1. For the exterior derivative we thus obtain,

$$\mathrm{d}\rho = \frac{\nu}{2}\rho_\alpha(\cdot, \mathsf{K}) \wedge \rho_\alpha = \nu \iota_{\mathsf{K}} \Omega\,, \tag{2.11}$$

proving the proposition. □

The **moment map** associated to the Killing vector K is the section $P = P^\alpha J_\alpha \in \Gamma(Q)$ related to the two form ρ by $\rho = \nu g(\cdot, P)$, cf. [18]. It follows that

$$P^\alpha = \epsilon\langle \nabla \mathsf{K}, J_\alpha \rangle\,, \quad \epsilon := \tfrac{1}{\nu}\,. \tag{2.12}$$

This is consistent with the use of P^α in (2.6). We are interested in the gradient flow generated by the function

$$f := P^2 := \langle P, P \rangle = \epsilon^2 \sum \langle \nabla \mathsf{K}, J_\alpha \rangle^2\,, \tag{2.13}$$

which we call the **energy**.

Proposition 2 *The covariant derivative of the moment map P is given by*

$$\nabla P = \tfrac{1}{2}\rho_\alpha(\cdot, \mathsf{K}) \otimes J_\alpha\,. \tag{2.14}$$

The gradient of the energy is

$$\operatorname{grad} f = P\mathsf{K} = P^\alpha J_\alpha \mathsf{K} = \epsilon\langle \nabla \mathsf{K}, J_\alpha \rangle J_\alpha \mathsf{K}\,. \tag{2.15}$$

Proof: The formula (2.14) is an immediate consequence of (2.9). Using (2.12) and (2.14) we compute the differential $\mathrm{d}f$:

$$\mathrm{d}f = 2\langle \nabla P, P \rangle = P^\alpha \rho_\alpha(\cdot, \mathsf{K}) = \epsilon\langle \nabla \mathsf{K}, J_\alpha \rangle \rho_\alpha(\cdot, \mathsf{K})\,. \tag{2.16}$$

This implies the formula for the gradient. □

Corollary 1 *The set of critical points of P is*

$$\mathrm{Crit}(P) = \{\mathsf{K} = 0\}\,. \tag{2.17}$$

The set of critical points of the energy f is the union

$$\mathrm{Crit}(f) = \{\mathsf{K} = 0\} \cup \{f = 0\}\,. \tag{2.18}$$

The formula (2.14) appears in supergravity as the definition of the moment map or 'pre-potential'[6] in the component form:

$$-\nu \mathcal{D}_X P^\alpha = \mathcal{R}^\alpha_{XY} \mathsf{K}^Y = \frac{\nu}{2} J^\alpha_{XY} \mathsf{K}^Y\,. \tag{2.19}$$

[6]Note that the form $\rho_\alpha(\cdot, \mathsf{K}) = g(\cdot, J_\alpha \mathsf{K})$ has components $-J^\alpha_{XY}\mathsf{K}^Y$.

In supergravity $\nu = -1$. Here,

$$\mathcal{R}^{\alpha}_{XY} = 2\partial_{[X}\omega^{\alpha}_{Y]} + 2\omega^{\beta}_{X}\omega^{\gamma}_{Y}\varepsilon_{\alpha\beta\gamma}\,, \tag{2.20}$$

are the components of the $\mathfrak{sp}(1)$ curvature (2.10). They are clearly proportional to the Kähler forms ρ_α, yielding the formula for the gradient

$$\partial_X f = -P^{\alpha} J^{\alpha}_{XY}\mathsf{K}^{Y}\,. \tag{2.21}$$

Remark: The set $\{\mathsf{K} = 0\}$ is a union of *totally geodesic* submanifolds. This follows from the fact that a connected component of the fixed point set of a group of isometries is totally geodesic since isometries transform geodesics to geodesics and there exists a unique geodesic through two sufficiently close points. If (M, g) is complete and has non-positive sectional curvature (e.g. if (M, g) is a symmetric space of non-compact type or, more generally, a Riemannian manifold covered by such a space) then $\{\mathsf{K} = 0\}$ is *connected* since in the universal covering of M any two points are joint by a unique geodesic.

This generalizes to any symmetric space allowed in supergravity (which has to be non-compact due to the $\nu = -1$ condition) the result found in the toy model (universal hypermultiplet) in [7]. Namely, if there is an isolated critical point, then there are no other critical points, or, if there are two critical points then, as explained above, they are connected by a geodesic which consists of critical points.

3 The Hessian of the energy at a critical point

In this section we compute the Hessian of the energy f at critical points and study its spectrum. For this we need the following lemma.

Lemma 2 *The second covariant derivative of the moment map is given by:*

$$\nabla^2_{\mathsf{X},\mathsf{Y}} P = \tfrac{1}{2}\rho_\alpha(\mathsf{Y}, \nabla_{\mathsf{X}}\mathsf{K})J_\alpha\,. \tag{3.1}$$

The Hessian of the energy is:

$$\mathrm{Hess}_f(\mathsf{X},\mathsf{Y}) := \nabla^2_{\mathsf{X},\mathsf{Y}} f = P^{\alpha}\rho_\alpha(\mathsf{Y}, \nabla_{\mathsf{X}}\mathsf{K}) + \tfrac{1}{2}\rho_\alpha(\mathsf{X},\mathsf{K})\rho_\alpha(\mathsf{Y},\mathsf{K})\,. \tag{3.2}$$

Proof: Using (2.14) we compute

$$\begin{aligned}
\nabla^2_{\mathsf{X},\mathsf{Y}} P &= \tfrac{1}{2}g(\mathsf{Y}, \nabla_{\mathsf{X}}(J_\alpha)\mathsf{K})J_\alpha + \tfrac{1}{2}g(\mathsf{Y}, J_\alpha\nabla_{\mathsf{X}}\mathsf{K})J_\alpha + \tfrac{1}{2}g(\mathsf{Y}, J_\alpha\mathsf{K})\nabla_{\mathsf{X}}J_\alpha \\
&= \tfrac{1}{2}\rho_\alpha(\mathsf{Y}, \nabla_{\mathsf{X}}\mathsf{K})J_\alpha - \varepsilon_{\alpha\beta\gamma}\left[\omega_\beta(\mathsf{X})g(\mathsf{Y}, J_\gamma\mathsf{K})J_\alpha + g(\mathsf{Y}, J_\alpha\mathsf{K})\omega_\beta(\mathsf{X})J_\gamma\right] \\
&= \tfrac{1}{2}\rho_\alpha(\mathsf{Y}, \nabla_{\mathsf{X}}\mathsf{K})J_\alpha\,.
\end{aligned}$$

For the Hessian of $f = P^2$ we get:

$$\begin{aligned}
\nabla^2_{\mathsf{X},\mathsf{Y}} f &= 2\langle\nabla^2_{\mathsf{X},\mathsf{Y}}P, P\rangle + 2\langle\nabla_{\mathsf{X}}P, \nabla_{\mathsf{Y}}P\rangle \\
&= \epsilon\rho_\alpha(\mathsf{Y}, \nabla_{\mathsf{X}}\mathsf{K})\langle\nabla\mathsf{K}, J_\alpha\rangle + \tfrac{1}{2}\rho_\alpha(\mathsf{X},\mathsf{K})\rho_\alpha(\mathsf{Y},\mathsf{K}) \\
&= P^{\alpha}\rho_\alpha(\mathsf{Y}, \nabla_{\mathsf{X}}\mathsf{K}) + \tfrac{1}{2}\rho_\alpha(\mathsf{X},\mathsf{K})\rho_\alpha(\mathsf{Y},\mathsf{K})
\end{aligned}$$

□

Let us decompose the operator

$$L_{\mathsf{K}} := \nabla\mathsf{K} = \langle \nabla\mathsf{K}, J_\alpha\rangle J_\alpha + \bar{L}_{\mathsf{K}} = \nu P^\alpha J_\alpha + \bar{L}_{\mathsf{K}}\,. \tag{3.3}$$

Then $\bar{L}_{\mathsf{K}}$ is a skew symmetric operator commuting with Q. This follows from the fact that ∇ and K preserve Q using the formula

$$\nabla\mathsf{K} = \nabla_{\mathsf{K}} - \mathcal{L}_{\mathsf{K}}\,. \tag{3.4}$$

The operators $\langle \nabla\mathsf{K}, J_\alpha\rangle J_\alpha$ and $\bar{L}_{\mathsf{K}}$ are called the $\mathfrak{sp}(1)$-part and the $\mathfrak{sp}(r)$-part of L_{K}, respectively. The important properties of $\bar{L}_{\mathsf{K}}$ are

$$(S_{\mathsf{K}})_{\alpha XY} := J_{\alpha X}{}^Z \left(\bar{L}_{\mathsf{K}}\right)_{ZY} = (S_{\mathsf{K}})_{\alpha YX}\ , \qquad (S_{\mathsf{K}})_{\alpha X}{}^X = 0\,. \tag{3.5}$$

Theorem 2 *The set of critical points of the energy f is given in (2.18). At a point $p \in M$ where $f = 0$ the Hessian of f is given by*

$$\mathrm{Hess}_f(\mathsf{X},\mathsf{X}) = \tfrac{1}{2}\sum \rho_\alpha(\mathsf{X},\mathsf{K})^2 \tag{3.6}$$

and hence it is positive semi-definite. Its kernel is

$$\ker \mathrm{Hess}_f = \mathrm{span}\{J_1\mathsf{K}, J_2\mathsf{K}, J_3\mathsf{K}\}^\perp \subset T_pM\,. \tag{3.7}$$

At a point where $\mathsf{K} = 0$ the Hessian is given by

$$\begin{aligned} \mathrm{Hess}_f(\mathsf{X},\mathsf{X}) &= P^\alpha \rho_\alpha(\mathsf{X}, \nabla_{\mathsf{X}}\mathsf{K}) \\ &= -\nu f g(\mathsf{X},\mathsf{X}) + g(\mathsf{X}, S\mathsf{X})\,, \end{aligned} \tag{3.8}$$

where S is the symmetric operator $S := P\bar{L}_{\mathsf{K}} = P^\alpha J_\alpha \bar{L}_{\mathsf{K}} = P^\alpha \left(S_{\mathsf{K}}\right)_\alpha$.

Proof: This follows immediately from (2.16), (3.2) and the decomposition of L_{K} into its $\mathfrak{sp}(1)$ and $\mathfrak{sp}(r)$-parts. □

As this is a main result of this section, we give also its component form:

$$\partial_X \partial_Y f|_{\mathsf{K}=0} = -\nu\, f\, g_{XY} + P^\alpha \left(S_{\mathsf{K}}\right)_{\alpha XY}\ . \tag{3.9}$$

Recall that the Hesse operator H_f is defined by $g(\mathrm{H}_f\mathsf{X},\mathsf{Y}) = \mathrm{Hess}_f(\mathsf{X},\mathsf{Y})$. We are now in a position to prove:

Theorem 3 *At a point where $\mathsf{K} = 0$ there exists an eigenbasis for the Hesse operator H_f of the form*

$$e_1\,,\ J_1e_1\,,\ e_2\,,\ J_1e_2\,,\ldots,\ e_r\,,\ J_1e_r\,;\ J_2e_1\,,\ J_3e_1\,,\ J_2e_2\,,\ J_3e_2\,,\ldots,\ J_2e_r\,,\ J_3e_r\,.$$

The corresponding eigenvalues are

$$\lambda_1 - \nu f\,,\ \lambda_1 - \nu f\,,\ \lambda_2 - \nu f\,,\ \lambda_2 - \nu f\,,\ldots,\ \lambda_r - \nu f\,,\ \lambda_r - \nu f\,;$$

$$-\lambda_1 - \nu f\,,\ -\lambda_1 - \nu f\,,\ -\lambda_2 - \nu f\,,\ -\lambda_2 - \nu f\,,\ldots,\ -\lambda_r - \nu f\,,\ -\lambda_r - \nu f\,.$$

Proof: The eigenvectors of $\mathrm{H}_f = -\nu f\,\mathrm{id} + S$ coincide with the eigenvectors of the operator $S = P^\alpha J_\alpha \bar{L}_{\mathsf{K}}$. Without loss of generality we can assume that $P^\alpha J_\alpha$ is proportional to J_1: $P^\alpha J_\alpha = cJ_1$, $c \in \mathbb{R}$. Then the operator $S = P^\alpha J_\alpha \bar{L}_{\mathsf{K}} = cJ_1 \bar{L}_{\mathsf{K}}$ commutes with J_1 and anticommutes with J_2 and J_3. Let v be an eigenvector of S. Then

$$Sv = \lambda v \quad SJ_1 v = J_1 S v = \lambda J_1 v\,, \quad SJ_2 v = -J_2 S v = -\lambda J_2 v\,, \quad \text{and} \quad SJ_3 v = -\lambda J_3 v\,.$$

Now one can easily prove by induction that there is an eigenbasis of S of the form

$$e_1, J_1 e_1, e_2, J_1 e_2, \ldots, e_r, J_1 e_r, J_2 e_1, J_3 e_1, J_2 e_2, J_3 e_2, \ldots, J_2 e_r, J_3 e_r$$

with eigenvalues

$$\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_r, \lambda_r, -\lambda_1, -\lambda_1, \ldots, -\lambda_r, -\lambda_r\,.$$

□

We will say that the $\mathfrak{sp}(1)$-part of $\nabla\mathsf{K}$ is **small** (at a point where $\mathsf{K} = 0$) if $|\nu f| < |\lambda_i|$ for all i. We will say that the $\mathfrak{sp}(r)$-part is **regular** if $\bar{L}_{\mathsf{K}}$ is invertible. This is the case if and only if the $\lambda_i \neq 0$.

Corollary 2 *Let $p \in M$ be a point where $\mathsf{K} = 0$ and let the $\mathfrak{sp}(1)$-part of $\nabla\mathsf{K}$ be small (and therefore the $\mathfrak{sp}(r)$-part is regular). Then the Hessian Hess_f has r positive and r negative eigenvalues, each of double multiplicity.*

We recall that all the **known non-flat homogeneous quaternionic-Kähler manifolds** fall into two classes: the Wolf spaces [19] and the Alekseevsky spaces [20, 21, 22]. The Wolf spaces are symmetric spaces of positive scalar curvature. Their isometry group is compact. The Alekseevsky spaces are precisely the homogeneous quaternionic-Kähler manifolds of negative scalar curvature which admit an $\mathbb{R}$-splittable simply transitive solvable group of isometries. This class contains the non-compact duals of the Wolf spaces, which are symmetric spaces of negative scalar curvature, together with 3 series of non-symmetric quaternionic-Kähler manifolds. The following result characterizes the symmetric quaternionic-Kähler manifolds.

Theorem 4 *[23] A homogeneous quaternionic-Kähler manifold is symmetric if and only if it admits a smooth compact quotient by a discrete group of isometries.*

Theorem 5 *Let $(M = G/H, g, Q)$ be one of the known non-flat homogeneous quaternionic-Kähler manifolds. Then there exists a Killing vector field K vanishing at a point $p \in M$ such that the Hessian Hess_f has split signature at p.*

Proof: By Corollary 2 it is sufficient to prove that the isotropy Lie algebra $\mathfrak{h}$ contains a vector with small $\mathfrak{sp}(1)$-part and regular $\mathfrak{sp}(r)$-part. For the symmetric quaternionic-Kähler manifolds the isotropy Lie algebra splits as a direct sum of ideals:

$$\mathfrak{h} = \mathfrak{sp}(1) \oplus \mathfrak{h}'\,, \quad \mathfrak{h}' \subset \mathfrak{sp}(r)\,. \tag{3.10}$$

(For most of the symmetric quaternionic-Kähler manifolds $\mathfrak{h}' = [\mathfrak{h}', \mathfrak{h}']$. The only exception is when G is of type A_{n+1}, with $\mathfrak{h}' = \mathfrak{u}(n)$ and $[\mathfrak{u}(n), \mathfrak{u}(n)] = \mathfrak{su}(n)$). Let $\mathfrak{t} \subset \mathfrak{h}'$ be a Cartan subalgebra and $T \in \mathfrak{t}$ a regular element. Then T has regular $\mathfrak{sp}(r)$-part (has only non-zero eigenvalues under the isotropy representation) since the isotropy representation of $\mathfrak{h}'$ has no trivial submodule. By adding a small vector from $\mathfrak{sp}(1)$ we obtain a Killing vector with the desired properties.

The non-symmetric case is more involved since the isotropy group $\mathfrak{h} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(r)$ does not admit a splitting of the type (3.10). The isometry and isotropy groups of these spaces were found in [24] (see also summary in [25]). We use the description of the Alekseevsky spaces given in [26]. This does not use the *full* isometry group. The metric-preserving group in the centralizer of the Clifford algebra, which consists of the antisymmetric matrices commuting with the gamma matrices, is not included. This group is part of the full isometry and the full isotropy group.

Let $M = G/H$ be an Alekseevsky space of dimension $4r$. The Lie group $G = G(\Pi)$ is defined by a $\mathfrak{spin}(V)$-equivariant map $\Pi : \wedge^2 W \to V$, where V is a pseudo-Euclidean vector space and W is a module for the even Clifford algebra $C\ell^0(V)$. The isotropy Lie algebra has the form[7]

$$\mathfrak{h} = \mathfrak{so}(3) \oplus \mathfrak{so}(p, q+3)\,, \tag{3.11}$$

where $(p+3, q+3)$ is the signature of V. Note that the $\mathfrak{so}(3)$ here is not the one that acts as the quaternionic structure, which we consistently denote as $\mathfrak{sp}(1)$. We denote by $\pi : \mathfrak{h} \to \mathfrak{sp}(r)$ the $\mathfrak{sp}(r)$-projection of $\mathfrak{h}$. It is faithful and has parts from both subalgebras of (3.11). The $\mathfrak{sp}(1)$-projection of $\mathfrak{h}$ has kernel $\mathfrak{so}(p, q+3)$ and defines an isomorphism $\mathfrak{so}(3) \to \mathfrak{sp}(1) = Q$. The isotropy module splits under $\pi(\mathfrak{h}) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(p, q+3)$ as follows

$$\mathbb{C}^2 \oplus \mathbb{C}^2 \otimes \mathbb{R}^{p,q+3} \oplus W\,. \tag{3.12}$$

Here $\mathfrak{so}(3)$ acts on $\mathbb{C}^2 = \mathbb{H}$ by the standard representation of $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ commuting with the quaternionic structure and $\mathfrak{so}(p, q+3)$ acts trivially on $\mathbb{C}^2$ and in standard way on $\mathbb{R}^{p,q+3}$. The action on the $C\ell^0(V)$-module W is induced by the inclusion

$$\mathfrak{so}(3) \oplus \mathfrak{so}(p, q+3) \subset \mathfrak{so}(V) = \mathfrak{so}(p+3, q+3) \cong \mathfrak{spin}(V) \subset C\ell^0(V)\,.$$

From this description we see that the isotropy module of $\pi(\mathfrak{h})$ has no trivial submodules. This proves that $\mathfrak{h}$ contains elements with regular $\mathfrak{sp}(r)$-part. Also it is easy to see that the $\mathfrak{sp}(1)$-part of such an element has to be non-zero (due to the submodule $\mathbb{C}^2$) and can be chosen to be arbitrarily small. □

For the symmetric quaternionic-Kähler manifolds we can prove the existence of Killing vector fields K vanishing at a point $p \in M$ such that Hess_f is definite at p (non-degenerate local extremum of f).

Theorem 6 *Let $(M = G/H, g, Q)$ be a symmetric quaternionic-Kähler manifold of reduced scalar curvature ν. If $\nu > 0$ (respectively, $\nu < 0$), there exists a Killing vector field* K

[7] The q here is in agreement with the notation in [20, 22, 21, 24, 25]. The P or $\dot{P}$ values in these papers determine the choice of the module W. In [26] a generalization is made to homogeneous spaces of non-positive signature. This is reflected in the extra parameter p, which is 0 in the positive-signature case.

vanishing at a point $p \in M$ such that the Hessian Hess_f *is negative (respectively, positive) definite, i.e. the energy f has a non-degenerate local maximum (respectively, minimum) at p.*

Proof: It follows from the decomposition 3.10 that there exist Killing vector fields K vanishing at a point $p \in M$ with zero $\mathfrak{sp}(r)$-part at p. For such a field K the Hesse operator of f at p is given by $\mathrm{H}_f = -\nu f\,\mathrm{id}$. So $\mathrm{H}_f < 0$ if $\nu > 0$ and $\mathrm{H}_f > 0$ if $\nu < 0$. (The same is true for any Killing vector with sufficiently small $\mathfrak{sp}(r)$-part.) □

Note that for the non-symmetric Alekseevsky spaces there are no non-zero Killing vector fields in the isotropy Lie algebra with zero $\mathfrak{sp}(r)$-part. Nevertheless we can prove the following result.

Theorem 7 *Let $(M = G/H, g, Q)$ be an Alekseevsky space. It is defined by a $\mathfrak{spin}(V)$-equivariant map $\Pi : \wedge^2 W \to V$, where $V = \mathbb{R}^{p+3,q+3}$ and W is a $C\ell^0(V)$-module. Then there exists a Killing vector field K vanishing at a point $p \in M$ such that the Hessian* Hess_f *is positive semi-definite at p. More precisely, the spectrum of* Hess_f *consists of three eigenvalues: $\lambda := -\nu f > 0$, of multiplicity $4(r-p-q+2)$, the eigenvalue 2λ, of multiplicity $2(p+q+4)$, and the eigenvalue 0, of multiplicity $2(p+q+4)$.*

Proof: It is sufficient to choose $\mathsf{K} \in \mathfrak{so}(3) \subset \mathfrak{h} = \mathfrak{so}(3) \oplus \mathfrak{so}(p, q+3)$. At the canonical base point we can assume without loss of generality that $P = \epsilon J_1$ (and hence $f = \epsilon^2$). Then the Hesse operator at the canonical base point is $\mathrm{H}_f = -\nu f\,\mathrm{id} + S = -\nu\epsilon^2\,\mathrm{id} + S = -\epsilon\,\mathrm{id} + S$, where $S = \epsilon J_1 \bar{L}_{\mathsf{K}}$. Recall that the decomposition (3.12) of the isotropy module splits under the $\mathfrak{sp}(r)$-projection $\pi(\mathfrak{h})$. With respect to that decomposition S acts trivially on W and acts only on the first factor $\mathbb{C}^2$ of $\mathbb{C}^2 \oplus \mathbb{C}^2 \otimes \mathbb{R}^{p,q+3} = \mathbb{C}^2 \otimes \mathbb{R}^{p,q+4}$ with eigenvalues $\pm\epsilon$ of double multiplicity. This shows that H_f has eigenvalues $-\epsilon$ (of multiplicity $\dim W$), -2ϵ (of multiplicity $2r - \dim W/2$) and 0 (of multiplicity $2r - \dim W/2$). □

In the next sections we want to extend our discussion to the manifolds which are allowed targets for the scalars of 5-dimensional supergravity theories. The most general such manifold for a theory with r hypermultiplets and n vector multiplets is a product $M \times N$ of a quaternionic-Kähler manifold of dimension $4r$ and a *very special manifold* N of dimension n.

4 Very special real manifolds

The geometry connected to vector multiplets in 5 dimensions was uncovered in an old beautiful paper [4]. The manifolds were placed in the context of the family of special geometries in [5].

A **very special manifold** is a connected immersed hypersurface $N \hookrightarrow \{C = 1\} \subset \mathbb{R}^{n+1}$ defined by a homogeneous cubic polynomial C which is non-singular on a neighborhood of the image of the immersion. For simplicity of our exposition, we assume, without loss of generality, that $N \subset \mathbb{R}^{n+1}$ is embedded. Then we do not need to distinguish between points of N and their images in $\mathbb{R}^{n+1}$. The radial vector field ξ defined by $\xi(p) = p$ is always transversal to the hypersurface N. It gives rise to a pseudo-Riemannian metric $g = g_N$ and to a torsionfree connection D on N. They are defined by the formula[8]

$$\partial_{\mathsf{X}}\mathsf{Y} = D_{\mathsf{X}}\mathsf{Y} + \tfrac{2}{3}g(\mathsf{X},\mathsf{Y})\xi\,. \tag{4.1}$$

Here X and Y are tangent to N and ∂ denotes the canonical connection of $\mathbb{R}^{n+1}$. Usually one assumes that the metric is positive definite.

Let us denote by $C(\cdot,\cdot,\cdot)$ the completely symmetric trilinear form whose associated cubic form is $C = C(\cdot)$. They are related by polarization: $C(\mathsf{X},\mathsf{X},\mathsf{X}) = C(\mathsf{X})$.

Proposition 3 *The metric g is related to the Hessian of C by*

$$g = -\tfrac{1}{2}\mathrm{Hess}_C|N\,. \tag{4.2}$$

More explicitly, for all $\mathsf{X},\mathsf{Y} \in T_pN$:

$$g(\mathsf{X},\mathsf{Y}) = -3C(p,\mathsf{X},\mathsf{Y})\,. \tag{4.3}$$

Proof: Equation (4.2) easily implies the explicit formula (4.3) since C is a homogeneous function of degree 3. Let X and Y be vector fields tangent to N. We can extend them (locally) to vector fields in the ambient space $\mathbb{R}^{n+1}$ satisfying $\mathsf{X}C = \mathsf{Y}C = 0$. Hence, using the homogeneity of C we obtain at a point $p \in \{C = 1\}$:

$$\mathrm{Hess}_C(\mathsf{X},\mathsf{Y}) = \mathsf{X}\mathsf{Y}C - \partial_{\mathsf{X}}\mathsf{Y}C = 0 - 3C(p,p,\partial_{\mathsf{X}}\mathsf{Y}) = -3C(p,p,\partial_{\mathsf{X}}\mathsf{Y})\,. \tag{4.4}$$

The right hand side of (4.4) equals $-2g(\mathsf{X},\mathsf{Y})$ because the linear form $C(p,p,\cdot)$ vanishes precisely on the tangent space of N at p and equals 1 on $p = \xi(p)$. $\square$

We define a $(1,2)$-tensor S^C on N by

$$g(S^C_{\mathsf{X}}\mathsf{Y},\mathsf{Z}) = \tfrac{3}{2}C(\mathsf{X},\mathsf{Y},\mathsf{Z}) \quad \text{for all} \quad \mathsf{X},\mathsf{Y},\mathsf{Z} \in T_pN\,. \tag{4.5}$$

Lemma 3 *The Levi-Civita connection of g is given by*

$$\nabla = D - S^C\,. \tag{4.6}$$

Proof: The connection $\nabla = D - S^C$ is torsionfree because D is torsionfree and $S^C_{\mathsf{X}}\mathsf{Y} = S^C_{\mathsf{Y}}\mathsf{X}$. We check that it is a metric connection. Let $\mathsf{X},\mathsf{Y},\mathsf{Z}$ be tangent vectors to N (locally) extended to vector fields in $\mathbb{R}^{n+1}$ satisfying $\mathsf{X}C = \mathsf{Y}C = \mathsf{Z}C = 0$. Then we compute

$$\begin{aligned}
(\nabla_{\mathsf{X}}g)(\mathsf{Y},\mathsf{Z}) &= \mathsf{X}g(\mathsf{Y},\mathsf{Z}) - g(\nabla_{\mathsf{X}}\mathsf{Y},\mathsf{Z}) - g(\mathsf{Y},\nabla_{\mathsf{X}}\mathsf{Z})\\
&= \mathsf{X}g(\mathsf{Y},\mathsf{Z}) - g(D_{\mathsf{X}}\mathsf{Y},\mathsf{Z}) - g(\mathsf{Y},D_{\mathsf{X}}\mathsf{Z}) + g(S^C_{\mathsf{X}}\mathsf{Y},\mathsf{Z}) + g(\mathsf{Y},S^C_{\mathsf{X}}\mathsf{Z})\\
&\overset{(4.1)}{=} \mathsf{X}g(\mathsf{Y},\mathsf{Z}) - g(\partial_{\mathsf{X}}\mathsf{Y},\mathsf{Z}) - g(\mathsf{Y},\partial_{\mathsf{X}}\mathsf{Z}) + g(S^C_{\mathsf{X}}\mathsf{Y},\mathsf{Z}) + g(\mathsf{Y},S^C_{\mathsf{X}}\mathsf{Z})\\
&= (\partial_{\mathsf{X}}g)(\mathsf{Y},\mathsf{Z}) + 2C(\mathsf{X},\mathsf{Y},\mathsf{Z})\\
&\overset{(4.2)}{=} -\tfrac{1}{3}(\partial_{\mathsf{X}}\mathrm{Hess}_C)(\mathsf{Y},\mathsf{Z}) + 2C(\mathsf{X},\mathsf{Y},\mathsf{Z})\\
&= -\tfrac{1}{3}(\partial^3C)(\mathsf{X},\mathsf{Y},\mathsf{Z}) + 3C(\mathsf{X},\mathsf{Y},\mathsf{Z})\\
&= -3C(\mathsf{X},\mathsf{Y},\mathsf{Z}) - 3C(\mathsf{X},\mathsf{Y},\mathsf{Z}) = 0\,.
\end{aligned}$$

[8]The factor 2/3 is introduced for consistency of the notation with the supergravity action. It guarantees that the corresponding scalars have the same normalization in the kinetic energy as the scalars of the quaternionic manifold. The translation of these formulae to the familiar supergravity language will be given at the end of this section.

□

It is customary to denote the standard coordinates of $\mathbb{R}^{n+1}$ by h^I, with $I = 0, 1, \ldots, n$, and the cubic polynomial is then $C = C_{IJK} h^I h^J h^K$. The conjugate coordinates are $h_I := C_{IJK} h^J h^K$. One may choose local coordinates ϕ^x with $x = 1, \ldots, n$ on the hypersurface N. Vector fields tangent to N are those Y's for which

$$\mathsf{Y} = \mathsf{Y}^I \partial_I = \mathsf{Y}^x \partial_x\,, \qquad \rightarrow \qquad \mathsf{Y}^I = \mathsf{Y}^x(\partial_x h^I)\,. \tag{4.7}$$

The equation (4.1) is then the decomposition of the derivative

$$\partial_x \left[\mathsf{Y}^y(\partial_y h^I)\right] = (D_x \mathsf{Y}^y)(\partial_y h^I) + h^I g_{xy} \mathsf{Y}^y\,, \tag{4.8}$$

and the orthogonality of h^I and $\partial_y h^I$ implies that this defines the metric g_{xy}. The lemma 3 corresponds to the equation (where the semi-colon indicates covariant differentiation w.r.t. ϕ^x using Christoffel connection calculated from the metric g_{xy}) [4]

$$(\partial_y h^I)_{;x} = -\sqrt{\tfrac{2}{3}} T_{xy}^z (\partial_z h^I) + h^I g_{xy}\,, \qquad T_{xyz} := \left(\tfrac{3}{2}\right)^{3/2} C_{IJK}(\partial_x h^I)(\partial_y h^J)(\partial_z h^K)\,, \tag{4.9}$$

such that the derivative D on a vector tangent to the hypersurface corresponds to

$$D_x \mathsf{Y}^y = \mathsf{Y}^y{}_{;x} - \sqrt{\tfrac{2}{3}} T_{xz}^y \mathsf{Y}^z\,. \tag{4.10}$$

5 The dressed moment map

Let (M, g_M, Q) be a quaternionic-Kähler manifold of dimension $4r$ and (N, g_N) a very special manifold of dimension n, $N \subset \mathbb{R}^{n+1}$. We denote by π_M and π_N the projections of the product $M \times N$ and by $g = \pi_M^* g_M + \pi_N^* g_N$ the product metric. We assume that we are given a Lie algebra γ spanned by $n+1$ Killing vector fields K_I, $I = 0, 1, \ldots, n$, on M, with the corresponding moment maps being $P_I : M \to Q$. We define the **dressed moment map** $P : M \times N \to Q$ by

$$P := P^\alpha J_\alpha := h^I P_I \qquad \text{or} \qquad P_{XY} := P^\alpha J_{\alpha\, XY}\,. \tag{5.1}$$

It is a section of the bundle $\pi_M^* Q$ over $M \times N$, where $\pi_M : M \times N \to M$ is the projection. Let us also define[9]

$$\mathsf{K} := h^I \mathsf{K}_I\,. \tag{5.2}$$

It is an N-dependent vector field on M, a section of $\pi_M^* TM$. We want to study the gradient flow generated by the **energy** function $f := P^2$.

Proposition 4 *The covariant derivative of the dressed moment map P is given by*

$$\nabla P = \mathrm{d}h^I \otimes P_I + h^I \nabla P_I = \mathrm{d}h^I \otimes P_I + \tfrac{1}{2}\rho_\alpha(\cdot, \mathsf{K}) \otimes J_\alpha\,. \tag{5.3}$$

[9] In sections 2 and 3 we considered one isometry whose Killing vector we denoted by K and whose moment map we denoted by P. This can be considered in the context of this section as the case of a trivial very special real manifold, i.e. $n = 0$, with h^I having only the component $h^0 = 1$.

The differential of the energy is

$$\mathrm{d}f = 2\langle \nabla P, P\rangle = 2\langle P_I, P\rangle \mathrm{d}h^I + P^\alpha \rho_\alpha(\cdot, \mathsf{K})\,. \tag{5.4}$$

The gradient of the energy is

$$\operatorname{grad} f = 2\langle P_I, P\rangle \operatorname{grad} h^I + P\mathsf{K}\,. \tag{5.5}$$

Proof: This follows easily from Proposition 2 applied to the P_I. □

Corollary 3 *The set of critical points of P is [7]*

$$\mathrm{Crit}(P) = \{\mathrm{d}h^I \otimes P_I = 0\} \cap \{\mathsf{K} = 0\} = \{P_I = h_I P\} \cap \{\mathsf{K} = 0\}\,. \tag{5.6}$$

The set of critical points of the energy f is

$$\begin{aligned} \mathrm{Crit}(f) &= \Big\{\langle P, P_I\rangle \mathrm{d}h^I = 0\Big\} \cap \Big(\{\mathsf{K} = 0\} \cup \{f = 0\}\Big) \\ &= \Big\{\langle P, P_I\rangle = f h_I\Big\} \cap \Big(\{\mathsf{K} = 0\} \cup \{f = 0\}\Big)\,. \end{aligned} \tag{5.7}$$

In the supergravity context, critical points are points with constant scalars for solutions that preserve supersymmetry. The preservation of supersymmetry imposes for the N-sector the condition $P_I = h_I P$, i.e. it are the critical points of P.

6 The Hessian of the energy

In this section we carry over the calculations from section 3 to case of the dressed moment map.

Lemma 4 *The second covariant derivative of the dressed moment map is given by:*

$$\begin{aligned} \nabla^2_{\mathsf{X},\mathsf{Y}} P &= \mathrm{Hess}_{h^I}(\mathsf{X},\mathsf{Y}) P_I + \tfrac{1}{2}\rho_\alpha(\mathsf{X},\mathsf{K}_I)\mathsf{Y}(h^I) J_\alpha + \tfrac{1}{2}\mathsf{X}(h^I)\rho_\alpha(\mathsf{Y},\mathsf{K}_I) J_\alpha \\ &\quad + \tfrac{1}{2} h^I \rho_\alpha(\mathsf{Y}, \nabla_{\mathsf{X}}\mathsf{K}_I) J_\alpha\,. \end{aligned} \tag{6.1}$$

The Hessian of the energy is:

$$\begin{aligned} \mathrm{Hess}_f(\mathsf{X},\mathsf{Y}) &= 2\langle P, P_I\rangle \mathrm{Hess}_{h^I}(\mathsf{X},\mathsf{Y}) + P^\alpha \rho_\alpha(\mathsf{X},\mathsf{K}_I)\mathsf{Y}(h^I) \\ &\quad + P^\alpha \mathsf{X}(h^I)\rho_\alpha(\mathsf{Y},\mathsf{K}_I) + P^\alpha h^I \rho_\alpha(\mathsf{Y}, \nabla_{\mathsf{X}}\mathsf{K}_I) \\ &\quad + 2\langle \nabla_{\mathsf{X}} P, \nabla_{\mathsf{Y}} P\rangle. \end{aligned} \tag{6.2}$$

Proof: Let us compute from (5.3):

$$\nabla^2 P = (\nabla \mathrm{d}h^I) \otimes P_I + \nabla P_I \otimes \mathrm{d}h^I + \mathrm{d}h^I \otimes \nabla P_I + h^I \nabla^2 P_I\,. \tag{6.3}$$

This finishes the proof in view of Lemma 2. □

We put $\bar{L}_{\mathsf{K}} := h^I \bar{L}_{\mathsf{K}_I}$.

Theorem 8 *The Hessian of the energy is given by*

$$\begin{aligned}\mathrm{Hess}_f &= \tfrac{4}{3} f \pi_N^* g_N + 2\langle P, P_I\rangle \mathrm{d}h^I S^C \\ &\quad + \mathrm{d}h^I \otimes gP\mathsf{K}_I + gP\mathsf{K}_I \otimes \mathrm{d}h^I \\ &\quad + \pi_M^* g_M(-\nu f \mathrm{id} + S) + \nabla P \otimes \nabla P\,,\end{aligned}$$

where S^C is the $(1,2)$-tensor defined by (4.5) and S is the symmetric operator $S := P\bar{L}_{\mathsf{K}}$, $gA = g(A\cdot,\cdot)$ for an endomorphism A and $g\mathsf{X} = g(\mathsf{X},\cdot)$ for a vector X.

Proof: Using Lemma 4 we obtain

$$\begin{aligned}\mathrm{Hess}_f &= 2\langle P, P_I\rangle \mathrm{Hess}_{h^I} + \mathrm{d}h^I \otimes gP\mathsf{K}_I + gP\mathsf{K}_I \otimes \mathrm{d}h^I + h^I gP\nabla \mathsf{K}_I \\ &\quad + 2\langle \nabla P \otimes \nabla P\rangle\,.\end{aligned}$$

The decomposition (see (3.3))

$$\nabla \mathsf{K}_I = \nu P_I + \bar{L}_{\mathsf{K}_I} \tag{6.4}$$

shows that

$$h^I gP\nabla \mathsf{K}_I = -\nu f \pi_M^* g_M + gS\,. \tag{6.5}$$

To simplify the first term we need to calculate the Hessian Hess_{h^I} of the function $h^I|N$ with respect to the Levi-Civita connection ∇ of N. Of course, the Hessian $\partial^2 h^I$ of the linear function h^I with respect to the standard connection of $\mathbb{R}^{n+1}$ is zero. This means that $\mathsf{XY}h^I = (\partial_{\mathsf{X}}\mathsf{Y})h^I$ for all vector fields X and Y. So, using (4.1) and Lemma 3, we get for all vector fields X and Y tangent to N:

$$\begin{aligned}\mathrm{Hess}_{h^I}(\mathsf{X},\mathsf{Y}) &= \mathsf{XY}h^I - (\nabla_{\mathsf{X}}\mathsf{Y})h^I = (\partial_{\mathsf{X}}\mathsf{Y} - \nabla_{\mathsf{X}}\mathsf{Y})h^I \\ &= (D_{\mathsf{X}}\mathsf{Y} + \tfrac{2}{3} g_N(\mathsf{X},\mathsf{Y})\xi - \nabla_{\mathsf{X}}\mathsf{Y})h^I = \tfrac{2}{3} g_N(\mathsf{X},\mathsf{Y})h^I + (S^C_{\mathsf{X}}\mathsf{Y})h^I\,.\end{aligned}$$

□

In components the Hessian of h^I follows from (4.9). The expression $gP\mathsf{K}_I$ is the one-form with components

$$(gP\mathsf{K}_I)_X = -P_{XY}\mathsf{K}_I^Y\,. \tag{6.6}$$

On the other hand, the components of S (symmetric and traceless) are

$$S_{XY} = P^\alpha J_{\alpha X}{}^Z \left(\bar{L}_{\mathsf{K}}\right)_{ZY}\,. \tag{6.7}$$

The result of the theorem can therefore be written as

$$\begin{aligned}\partial_x\partial_y f &= \tfrac{4}{3}\left(f g_{xy} - \sqrt{\tfrac{2}{3}} P^\alpha P_I^\alpha T_{xy}^z \partial_z h^I\right),\\ \partial_x \partial_X f &= -P_{XY}\mathsf{K}_I^Y \partial_x h^I\,,\\ \partial_X\partial_Y f &= -\nu g_{XY} f + S_{XY}\,.\end{aligned} \tag{6.8}$$

Corollary 4 *At a critical point of P the Hesse operator of the energy, defined as*

$$\mathrm{H}_f(\cdot, g\cdot) = \mathrm{Hess}_f(\cdot,\cdot)\,, \tag{6.9}$$

is given by

$$\mathrm{H}_f = \tfrac{4}{3} f\pi_{N*} + \mathrm{d}h^I \otimes P\mathsf{K}_I + gP\mathsf{K}_I \otimes \operatorname{grad} h^I + (-\nu f \mathrm{id} + S)\circ \pi_{M*}\,. \tag{6.10}$$

Here π_{M*} and π_{N*} are the differentials of the canonical projections.

Proof: It suffices to remark that at a critical point $\nabla P{=}0$ and $P_I \mathrm{d}h^I{=}0$, see Corollary 3. □

7 Conclusions

We have considered the properties of flows governed by (1.2) and (1.4). In particular, we have obtained a formula for the Hessian matrix $\mathcal{U}$ defined in (1.5):

$$\mathcal{U} := \frac{3}{2f}\mathrm{H}_f\Big|_{\partial f=0} = \begin{pmatrix} 2\delta_x{}^y & +\frac{1}{W^2}(\partial_x \mathsf{K}^Z)P_Z{}^Y \\ -\frac{1}{W^2}P_{XZ}\partial^y\mathsf{K}^Z & -\nu\frac{3}{2}\delta_X{}^Y + \frac{1}{W^2}P_X{}^Z\left(\bar{L}_{\mathsf{K}}\right)_Z{}^Y \end{pmatrix}, \tag{7.1}$$

where the first entries are for the vector multiplets and the second for the hypermultiplets. ν is the reduced scalar curvature, see theorem 1, which is -1 in supergravity. P and $\bar{L}_{\mathsf{K}}$ select respectively the $\mathfrak{sp}(1)$ and $\mathfrak{sp}(r)$ parts of the dressed gauged isometry, defined by (6.6) and

$$\mathcal{D}_X\mathsf{K}_Y = \nu P_{XY} + \left(\bar{L}_{\mathsf{K}}\right)_{XY}\,. \tag{7.2}$$

Thus, in comparison with [7, 15], P is $-\mathcal{J}$ and $\bar{L}_{\mathsf{K}}$ is $\mathcal{L}$.

Note that if only vector multiplets are present, we have only the upper-left entry of (7.1), and thus only positive eigenvalues (UV attractors) [27, 28]. However, including the quaternionic-Kähler manifold (hypermultiplets) opens the possibility of having negative eigenvalues as well [11, 7]. The formula (7.1) implies that such eigenvalues are only possible if either the gauged isometries are mainly in the $\mathfrak{sp}(r)$ direction ($\bar{L}_{\mathsf{K}}$ big enough) or one gauges generators that are not in the isotropy group of the fixed point (such that $\mathsf{K}_I^X \neq 0$ and the off-diagonal elements are non-zero).

The lower-right $4r\times 4r$ part of (7.1) has eigenvalues

$$-\nu\left(\tfrac{3}{2}+\lambda_1\,,\,\tfrac{3}{2}+\lambda_1\,,\,\tfrac{3}{2}-\lambda_1\,,\,\tfrac{3}{2}-\lambda_1\,,\,\tfrac{3}{2}+\lambda_2\,,\,\tfrac{3}{2}+\lambda_2\,,\,\tfrac{3}{2}-\lambda_2\,,\,\tfrac{3}{2}-\lambda_2\,,\ldots,\,\tfrac{3}{2}-\lambda_r\right). \tag{7.3}$$

In various situations with trivial very special real manifold, we have found more detailed results on the structure of the eigenvalues, and the number of possible critical points. In particular, for complete quaternionic-Kähler manifolds of non-positive sectional curvature (which include the locally symmetric quaternionic-Kähler manifolds of negative scalar curvature) we find that the fixed point set, if non-empty, is either a point or a connected totally geodesic submanifold of non-zero dimension. Note that here, the fixed point set is non-empty if and only if K is a compact generator, i.e. if the closure of the group generated by K is compact. For homogeneous quaternionic-Kähler manifolds we prove the existence of Killing fields K such that the spectrum of $\mathcal{U}$ has some specific UV/IR-properties, for example:

(i) For all known quaternionic-Kähler manifolds we exhibit a Killing vector field K vanishing at a point $p \in M$ such that $\mathcal{U}$ has split signature at p.

(ii) For the symmetric quaternionic-Kähler manifolds of positive scalar curvature we find a Killing vector field K vanishing at a point $p \in M$ such that $\mathcal{U}$ is negative definite, i.e. the energy f has a non-degenerate local maximum at p.

(iii) For the symmetric quaternionic-Kähler manifolds of negative scalar curvature we find a Killing vector field K vanishing at a point $p \in M$ such that $\mathcal{U}$ is positive definite, i.e. the energy f has a non-degenerate local minimum at p.

(iv) For the known non-symmetric homogeneous quaternionic-Kähler manifolds (Alekseevsky spaces), which have negative scalar curvature, we find a Killing vector field K vanishing at a point $p \in M$ such that $\mathcal{U}$ is positive semi-definite, i.e. the energy f has a degenerate local minimum at p. Moreover we calculate the eigenvalues of $\mathcal{U}$.

We remark that supergravity selects negative scalar curvature ($\nu = -1$), so situation (ii) never occurs. We thus have either complete UV fixed points as in (iii), or zero directions as in (iv), or split signatures as in (i).

These results will be useful for the investigation of possibilities for flow lines from IR to UV critical points, suitable for AdS/CFT dual pairs, or from IR to IR critical points, relevant for the investigation of possible smooth supersymmetric domain-wall solutions. We also expect that the main results of this paper can also be applied to other dimensions where quaternionic-Kähler manifolds occur, e.g. 4-dimensional $N = 2$ supergravity theories.

A Remarks on the notation

Throughout this paper $Sp(n)$ denotes the compact real form of the symplectic group in $2n$ variables, which is sometimes denoted as $USp(2n)$. In particular $Sp(1) = USp(2) = SU(2)$.

Generically, $\mathcal{D}$ is a derivative that is covariant under all existing local symmetries. In other words, it is a connection in all bundles that are active on the field on which $\mathcal{D}$ acts. This means that $\mathcal{D}$ is an extension of the Levi-Civita connection (when acting on tangent vectors) adding a term $-$ (gauge field one-form) $\times$ gauge transformation, for any gauge transformation under which the object transforms. E.g. the latter gives rise to the last term in (2.3), containing the $\mathfrak{sp}(1)$ gauge field ω_α as it acts there on an object that transforms under $\mathfrak{sp}(1)$. The equation (2.3) determines ω_α and gives our convention for the $\mathfrak{sp}(1)$ transformation on triplets.

The curvature tensor of the Levi-Civita connection ∇ is defined by (denoting vector fields X in a coordinate basis as $\mathsf{X}^\Lambda \partial_\Lambda$)

$$\begin{aligned} \nabla_{\mathsf{X}} \nabla_{\mathsf{Y}} \mathsf{Z} - \nabla_{\mathsf{Y}} \nabla_{\mathsf{X}} \mathsf{Z} &= \nabla_{[\mathsf{X},\mathsf{Y}]} \mathsf{Z} + R(\mathsf{X},\mathsf{Y})\mathsf{Z}\,, \\ \langle R(\mathsf{X},\mathsf{Y})\mathsf{Z}, \mathsf{W}\rangle &= R^{\Upsilon}{}_{\Xi\Lambda\Sigma} \mathsf{X}^\Lambda \mathsf{Y}^\Sigma \mathsf{Z}^\Xi \mathsf{W}_\Upsilon\,. \end{aligned} \tag{A.1}$$

∇^2 is defined as

$$\nabla^2_{\mathsf{X},\mathsf{Y}} \equiv \nabla_{\mathsf{X}} \nabla_{\mathsf{Y}} - \nabla_{\nabla_{\mathsf{X}} \mathsf{Y}}\,. \tag{A.2}$$

The curvature of a connection $\mathcal{D}$ acting on tangent scalars in a space whose components are denoted by indices $\Lambda, \Xi, \ldots$ is given by

$$[\mathcal{D}_\Lambda, \mathcal{D}_\Xi] = -\mathcal{R}^a_{\Lambda\Xi} T_a \tag{A.3}$$

for any gauge symmetry denoted by indices a, and whose action is indicated here by T_a. See (2.20) for the $\mathfrak{sp}(1)$ curvature.

Acting on a vector field X with components X^Ξ in a local coordinate basis (neutral under other gauge transformations), $\nabla_{\mathsf{Y}} \mathsf{X} = \mathsf{Y}^\Lambda (\mathcal{D}_\Lambda \mathsf{X}^\Sigma)\partial_\Sigma$, and the curvature components of the Levi-Civita connection are given by

$$\mathcal{D}_\Lambda \mathcal{D}_\Sigma \mathsf{X}^\Xi - \mathcal{D}_\Sigma \mathcal{D}_\Lambda \mathsf{X}^\Xi = R^{\Upsilon}{}_{\Xi\Lambda\Sigma} \mathsf{X}^\Xi\,. \tag{A.4}$$

The Ricci tensor and scalar curvature are

$$Ric_{\Lambda\Sigma} = R^{\Xi}{}_{\Lambda\Xi\Sigma}\,, \qquad scal = g^{\Lambda\Sigma} Ric_{\Lambda\Sigma}\,. \tag{A.5}$$

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